THE THEORY OF RECEIVERS FOR SOUND IN WATER.

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SYNOPSIS.

Sound receivers in water; mathematical theory. (1) *Single receivers.* After proving that a receiver must have a small volume to be efficient, the author considers in detail the case of a small spherical receiver connected to the ear by a cylindrical tube. Its resonance frequency is independent of the size of the tube and depends only on the elasticity of the receiver and its effective mass. The sharpness of resonance, however, is greater for larger tubes and for smaller surfaces. The energy transmitted for a given pitch is a maximum for a certain tube size; and while it is independent of the surface at the resonance frequency, for lower pitches the intensity increases with the surface. It is shown that a resonating receiver with a properly chosen tube may concentrate into the tube energy from an area about 850 times the cross section of the tube. The theory agrees with experimental facts and leads to practical suggestions for the design of efficient receivers. (2) *Two receivers as used for direction finding.* A pair of receivers mounted on a horizontal rod which may be rotated about a vertical axis is an efficient device for getting the direction of a source of sound since it makes use of the binaural effect. The equations for the energy in each tube and for the difference in phase are derived. Unless the separation is very small, the conditions for correct direction finding are always satisfied. The best distance apart is half the wave-length in water for the resonance frequency of the receivers. (3) *Multiple receivers.* If n receivers are each connected by a tube with a cross section α to a tube whose cross section is $n\alpha$, and if the lengths are such that all the sounds meet in phase, practically all the sound may be concentrated in a single tube. Equations are derived for the case of symmetrically arranged receivers, all equidistant from the source of sound. Except at the resonance frequency, the multiple receivers should be, in practice, more sensitive than any single receiver. (4) *Lines of receivers.* By a special compensating device the sound from such a set may be concentrated in a single tube. Equations are derived for the energy transmitted in this case. (5) *Receivers distributed over large areas.* With flexible diaphrams, complete transmission may be secured if the ratio of the tube section to the diaphram area is properly chosen. With stiff diaphrams this is approximately true only for the resonance frequency of the receivers. Finally, the conditions for complete transmission for a given frequency are determined for the case of small receivers mounted in a plane in front of a totally reflecting parallel surface.

THIS paper contains an account of the theory of the action of "Broca tubes" and similar receivers for sound in water when used either singly or in combinations of any number.

1. SINGLE RECEIVERS.

The original Broca tube was simply a stethoscope which could be immersed in water. It consisted of a flat circular metal box one of the circular sides of which was made of a thin metal plate. A tube fixed into the center of the opposite side, led to the ears of the observer. Such an apparatus is shown in Fig. 1. When immersed in water and acted on by sound waves, the flexible diaphragm is made to vibrate by the pressure variations in the water and these vibrations are communicated to the air in the box and produce sound waves in the tube.

The box and diaphragm may be replaced by a rubber tube closed at one end as shown in Fig. 2, or by a spherical rubber ball, Fig. 3.

If the linear dimensions of the receiver are small compared with the

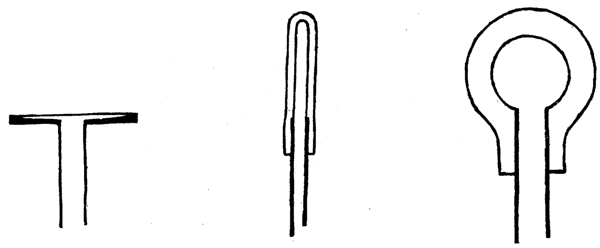

Fig. 1. Fig. 2. Fig. 3.

wave-length of the sound in air, then we can suppose that the pressure variation in the receiver is the same throughout its volume and is given by the equation

$$\delta P = -\gamma P \delta V / V,$$

where P denotes the pressure of the undisturbed air, δP the pressure change due to the sound, V the volume of the receiver, δV the volume variation due to the sound and γ the ratio of the specific heat of the air at constant pressure to that at constant volume.

If we suppose that $\delta V_1 = B\epsilon^{ipt}$ where δV_1 denotes the volume variation due to the motion of the diaphragm, B denotes the maximum value of δV_1, t denotes the time and $p = 2\pi n$, where n is the frequency, then we can easily calculate the amount of sound energy which passes up the tube leading from the receiver.

Let the displacement of the air in the tube from its undisturbed position be denoted by ξ and let

$$\xi = A\epsilon^{i(pt - k_1 x)}, \tag{1}$$

where x denotes the distance along the tube from the receiver and $k_1 = p/v_1$ where v_1 is the velocity of sound in air. Then at the junction between the receiver and the tube, the air pressure in the receiver must be equal to the air pressure in the tube so that at $x = 0$

$$\delta P = -\gamma P \delta V / V = -\gamma P d\xi / dx. \tag{2}$$

Also we have, if α denotes the cross section of the tube

$$\delta V = \delta V_1 + \alpha \underset{x=0}{\xi}, \tag{3}$$

because the total variation of the volume of the air in the receiver is equal to the variation due to the motion of the diaphragm plus that due to the motion of the air at the end of the tube. The equations (1), (2), and (3) give

$$A = \frac{B\epsilon^{i\theta}}{\sqrt{\alpha^2 + k_1^2 V^2}} \qquad \text{where} \qquad \tan\theta = -k_1 V/\alpha.$$

The flow of energy along the tube therefore varies as

$$\alpha A^2 = \frac{B^2 \alpha}{\alpha^2 + k_1^2 V^2}.$$

For a given value of V this is a maximum when $\alpha = k_1 V$ but for a given value of α it is greater, the smaller the value of V.

It is clear that V should be made small; a conclusion amply confirmed by experience. Provided $k_1 V$ is small compared with α it may be neglected and we get $A = B/\alpha$ which shows that then the volume of air leaving the receiver along the tube at $x = 0$ is equal to the diminution of the volume due to the motion of the walls of the receiver. In most cases which occur in practice $k_1 V$ is smaller than α. For example, a standard $\frac{1}{4}$ inch rubber receiver has an interior volume of about one c.c. so that for a wave-length $\lambda = 60$ cms. $k_1 V$ is about 0.1 and α is about 0.3 sq.cm. Also the effective volume of a tubular rubber receiver is probably much less than the actual volume because a large part of its volume variation is due to variations in the length of the receiver rather than to changes in its diameter.

The theory of sound receivers is greatly simplified when $k_1 V$ is small compared with α so in the rest of this paper it will be assumed that this is the case and the diminution of the volume of the receiver will be put equal to the volume of air expelled from it as though the air in the receiver were an incompressible fluid.

The variation of the volume of the receiver is, of course, due to the pressure variations in the surrounding water. These pressure variations are partly due to the incident sound and partly to the sound emitted into the water by the receiver itself. In the case of a spherical receiver, the sound emitted can be easily calculated but in the case of cylindrical and circular receivers the calculation is complicated. It is easy to see that the action of a receiver depends mainly on its volume variation and that its shape makes little difference except in so far as it influences the volume variation. In what follows, the receiver will be supposed spherical, as in Fig. 3, so that the results obtained should be exact for spherical receivers and will only be approximately true for receivers of any other form.

An elastic sphere of radius a immersed in water and vibrating radially, will emit spherical sound waves in which the pressure variation is inversely as the distance r from the center of the sphere. If the radius of the sphere is very small compared with the wave-length of the sound in water, then

$$\delta P = -\frac{Ek^2a^2\xi\epsilon^{-ikr}}{r},$$

where δP denotes the pressure variation in the water due to the emitted spherical waves, E the bulk modulus of elasticity for water, $k = 2\pi/\lambda$ where λ is the wave-length of the sound in water and ξ denotes the radial displacement of the surface of the sphere.

This result may be obtained as follows: Let the velocity potential of the emitted waves be given by

$$\phi = Br^{-1}\epsilon^{i(pt-kr)}$$

and let

$$\xi = D\epsilon^{ipt};$$

then

$$\delta P = -\rho\frac{d\phi}{dt} = -\rho\frac{Bip}{r}\epsilon^{i(pt-kr)}.$$

Also

$$\dot{\xi} = \frac{d\phi}{dr}{}_{(r=a)} = -B\left(\frac{1}{a^2}+\frac{ik}{a}\right)\epsilon^{i(pt-kr)}$$

$$= Dip\epsilon^{ipt}.$$

Hence

$$\delta P = -\frac{\rho p^2a^2\xi\epsilon^{-ikr}\cdot\epsilon^{ika}}{r(1+ika)}.$$

Also $E = \rho v^2$ and when ka is very small $\epsilon^{ika} = 1 + ika$ so that we get the equation given above for δP.

We can now go on to calculate the intensity of the sound produced in the tube leading from the receiver in terms of the intensity of the incident sound in the water.

The vibration of the surface of the spherical receiver is due to the variation of the outside water pressure minus the variation of the inside air pressure. Also we may suppose that the motion of the surface is resisted by an elastic restoring force proportional to the displacement and by a viscous resistance proportional to the velocity.

Let the pressure variation in the water due to the incident sound be denoted by $A\epsilon^{ipt}$. The total pressure variation in the water at $r = a$

will then be $A\epsilon^{ipt} - Ek^2a\xi\epsilon^{-ika}$. Let the potential of the sound in the tube be given by $\phi = G\epsilon^{i(pt-k_1x)}$, where x is the distance from the receiver measured along the tube.

Then the air pressure in the receiver where $x = 0$ is equal to

$$-\rho_1\frac{d\phi}{dt} = -\rho_1 Gip\epsilon^{ipt},$$

where ρ_1 = density of air.

Let m denote the mass of the spherical receiver, a its radius when undisturbed by sound, ξ the radial displacement of the surface due to sound, μ the restoring force per unit radial displacement due to the elasticity of the sphere, R the internal viscous resistance to the motion per unit radial velocity. Then we have

$$m\ddot{\xi} + R\dot{\xi} + \mu\xi = -4\pi a^2(A\epsilon^{ipt} - Ek^2a\xi\epsilon^{-ika} + Gip\rho_1\epsilon^{ipt}). \qquad (4)$$

We have also at $x = 0$ assuming that the air inside the receiver behaves like an incompressible fluid

$$\alpha\frac{d\phi}{dx} = -4\pi a^2\dot{\xi},$$

so that putting $4\pi a^2 = s$ we get

$$\xi = \frac{\alpha G}{sv_1}\epsilon^{ipt}.$$

Substituting this for ξ in (4) and putting $\epsilon^{+ika} = 1 + ika$ we get

$$G = \frac{-s^2v_1\alpha^{-1}A\epsilon^{-i\theta}}{\sqrt{(\mu - p^2(m + s\rho a))^2 + p^2\left(R + \dfrac{\rho s^2p^2}{4\pi v} + \dfrac{\rho_1 s^2 v_1}{\alpha}\right)^2}}, \qquad (5)$$

where

$$\tan\theta = \frac{p\left(R + \dfrac{\rho s^2p^2}{4\pi} + \dfrac{\rho_1 s^2v_1}{\alpha}\right)}{\mu - p^2(m + s\rho a)}.$$

If the elastic sphere were vibrating in a vacuum under the action of a radial force $-A\epsilon^{ipt}$ per unit area we should have

$$m\ddot{\xi} + R\dot{\xi} + \mu\xi = -sA\epsilon^{ipt},$$

so that if $\xi = B\epsilon^{ipt}$ we get

$$B = \frac{-sA\epsilon^{-i\theta}}{\sqrt{(\mu - mp^2)^2 + R^2p^2}},$$

where

$$\tan\theta = \frac{Rp}{\mu - mp^2}.$$

Comparing this with (5), we see that the mass of the sphere forming the receiver in water may be regarded as increased by $s\rho a$ and the viscous resistance to its motion by $\rho s^2 p^2/4\pi v$ due to the water and by $\rho_1 s^2 v_1/\alpha$ due to the air inside it. The additional viscous resistance is thus proportional to the square of the surface of the receiver so that very small receivers will be highly resonant provided R is small. For materials such as rubber and brass, experiments indicate that R is negligible when p is greater than about 100. R may therefore be put equal to zero.

For small receivers for which the total viscous resistance is small the expression for G is a maximum approximately when $\mu - p^2(m + s\rho a) = 0$. Let p_1 denote the value of p for which this is the case so that when $p = p_1$ and $R = 0$

$$G = \frac{iv_1 A}{p_1\left(\dfrac{\alpha\rho p^2}{4\pi v} + \rho_1 v_1\right)}.$$

In this case G has a phase 90° behind that of the incident sound and is independent of the physical properties of the sphere, except as regards p_1 but it depends on α the cross section of the tube leading from the receiver.

The energy E of the sound going up the tube, in unit time, is given by

$$E = \tfrac{1}{2}\rho_1 v_1 k_1^2 G^2 \alpha = \frac{1}{2}\frac{\rho_1}{v_1} p^2 G^2 \alpha.$$

Hence

$$E = \frac{\tfrac{1}{2}\rho_1 v_1 p^2 s^4 \alpha^{-1} A^2}{(\mu - p^2(m + s\rho a))^2 + p^2\left(R + \dfrac{s^2\rho_1 v_1}{\alpha} + \dfrac{s^2\rho p^2}{4\pi v}\right)^2}.$$

Hence when $p = p_1$ and $R = 0$

$$E = \frac{\tfrac{1}{2}\rho_1 v_1 \alpha A^2}{\left(\dfrac{\alpha\rho p_1^2}{4\pi v} + \rho_1 v_1\right)^2}. \tag{6}$$

This shows that E is a maximum with respect to α when

$$\rho_1 v_1 = \alpha\rho p_1^2/4\pi v \qquad \text{or} \qquad \alpha = \rho_1 v_1 \lambda_1^2/\pi\rho v,$$

where λ_1 is the wave-length in water for the frequency $p_1/2\pi$. In this case

$$E = \frac{1}{8}\frac{\alpha A^2}{\rho_1 v_1} = \frac{A^2\lambda_1^2}{8\pi\rho v}.$$

The amount of sound energy transmitted up the tube from the receiver can be conveniently expressed in terms of the area of wave front in the incident sound through which an equal amount of sound energy flows. If β denotes this area then $E = A^2\beta/2\rho v$ so that when $p = p_1$ and α has the best value

$$\frac{1}{2}\frac{A^2\beta}{\rho v} = \frac{A^2\lambda_1^2}{8\pi\rho v}$$

or

$$\beta = \frac{\lambda_1^2}{4\pi}.$$

Also when $p = p_1$ and α has the best value

$$E = \frac{1}{8}\frac{\alpha A^2}{\rho_1 v_1} = \frac{1}{2}\frac{\beta A^2}{\rho v},$$

so that

$$\frac{\beta}{\alpha} = \frac{1}{4}\frac{\rho v}{\rho_1 v_1} = 853.$$

Thus it appears that a receiver at its resonance frequency, when the cross section of the tube leading from it has the best value for this frequency, can concentrate into the tube the sound from an area 853 times that of the cross section of the tube.

In the expression found for E for any value of α if we put $M = m + s\rho a$ and $\alpha_1 = 4\pi v \rho_1 v_1/\rho p_1^2$ we get

$$E = \frac{\frac{1}{2}(\rho_1 v_1)^{-1}\alpha A^2}{\dfrac{\alpha^2 M^2}{(\rho_1 v_1)^2 s^4}\left\{\dfrac{p_1^2 - p^2}{p}\right\}^2 + \left\{1 + \dfrac{\alpha}{\alpha_1}\dfrac{p^2}{p_1^2}\right\}^2}.$$

This expression is a maximum with respect to α when

$$\alpha = \left\{\frac{M^2}{s^4(\rho_1 v_1)^2}\left(\frac{p_1^2 - p^2}{p}\right)^2 + \frac{p^4}{\alpha_1^2 p_1^4}\right\}^{-\frac{1}{2}},$$

which gives $\alpha = \alpha_1$ when $p = p_1$.

As an example suppose $a = 2$ cms., $M = 134$ grams, $p_1 = 5{,}000$,

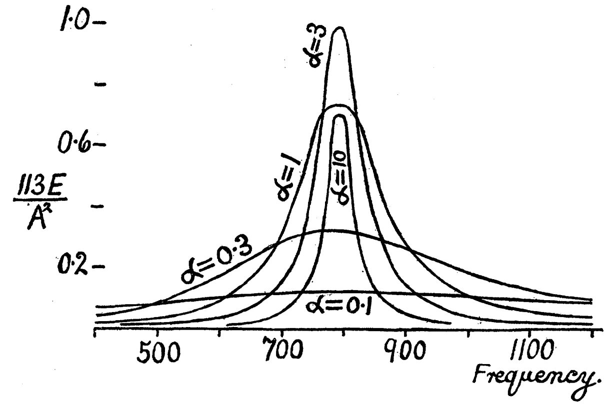

Fig. 4.

$R = 0$, $\alpha_1 = 3$ sq. cms. Fig. 4 shows the values of $113E/A^2$ in this case for five values of α, viz., 10, 3, 1, 0.3, and 0.1. It will be seen that increasing α makes the receiver more sharply resonant but diminishes the energy received both above and below the resonance frequency. This suggests a method of finding the resonance frequency of a receiver. If the receiver is connected to a wide tube of section much greater than α_1 it will be very insensitive except for frequencies near to its resonance frequency. The resonance frequency is not changed appreciably by varying α.

It is clear that if it is desired to make a receiver sensitive over a wide range of frequencies then the tube should be made with a section smaller than α_1. As the result of very many experiments with many types of receivers the following facts have been established when listening to a noise in the water like that from a boat or submarine:

1. The sound heard always has a more or less definite pitch characteristic of the receiver used.

2. Receivers having equal tubes and equal pitches give sounds of about equal intensity.

3. With receivers having equal tubes those of higher pitch give less intensity.

All the receivers for which the above results are true have small interior volumes.

Receivers with large volumes give less intensity than those with small volumes.

These results are in accordance with the theory. The sound from the boat is a noise having all frequencies between certain limits in its spectrum. The loudness falls off as the frequency increases. The receivers respond best to the frequencies near to p_1 for which the expression for E has its maximum value. This explains fact (1). Equation (6) shows that the sound energy obtained at the resonance frequency depends only on p_1 and α and that for a given value of α it decreases as p_1 increases. This agrees with facts (2) and (3).

It appears therefore that in the design of receivers, there are four things to which special attention should be given:

(1) Resonance frequency,
(2) Cross section of tube,
(3) Interior volume,
(4) Surface.

The resonance frequency determines the character of the sound received. Experience seems to show that it should be as high as is compatible with sufficient intensity. The best resonance frequency seems to be about 800 per sec. for many purposes.

The cross section of the tube should be equal to or rather less than $\alpha_1 = \rho_1 v_1 \lambda_1^2/\pi \rho v$ where λ_1 is the wave-length in the water at the resonance frequency. This gives about 3 sq. cms. for a frequency of 800 per sec. for which $\lambda_1 = 180$ cms. The interior volume of the receiver should be considerably less than $\alpha\lambda'/2\pi$ where λ' is the wave-length in air corresponding to the resonance frequency. If $\alpha = 3$ sq. cms. and $\lambda' = 43$ cms., then V should be much less than 20 c.c.; $V = 2$ c.c. would be quite small enough.

These conclusions apply to single receivers which when in use are not near to other receivers. The case of multiple receivers is considered in section (3) of this paper. As to surface, this has no influence at the resonance frequency but below the resonance frequency increasing the surface rapidly increases the intensity of the sound received. Receivers with small surfaces are sharply tuned while receivers with a large surface are sensitive over a wide range of frequencies.

When $p = p_1$ and $\alpha = \alpha_1$ we have seen that $E = \frac{1}{8}\alpha_1 A^2/\rho_1 v_1$ where A denotes the amplitude of the pressure variation in the water due to the incident sound. If Q denotes the amplitude of the pressure variation in the air in the tube leading from the receiver then $E = \frac{1}{2}\alpha_1 Q^2/\rho_1 v_1$ so that $Q = A/2$. Thus at the resonance frequency, when α has the best value for this frequency, the pressure variation in the air is one half the pressure variation in the water due to the incident sound.

2. Two Receivers, One Connected to Each Ear.

An important case is that in which two receivers are used, one connected to each ear by tubes of equal length. The two receivers are fixed on a horizontal line, in the water, at a constant distance apart. The two receivers can be rotated about a vertical axis bisecting the line joining them. The observer rotates the apparatus until the sound appears to come from a point directly in front of his head and then the source lies in the plane perpendicular to the line joining the two receivers and passing through its middle point. If the apparatus is turned slightly from this position the sound will appear to the observer to move towards the receiver which is moving nearer to the source. In this way the direction of the source in the horizontal plane can be accurately determined. This arrangement was proposed by the writer early in June, 1917, was developed by the General Electric Company and has been extensively employed during the war by the United States Navy.

The theory of this device can be worked out in the same way as that of a single receiver. The pressure variation on each receiver now includes a term due to the sound emitted by the other receiver. It will be supposed that the two receivers are exactly similar spheres and that their interior volume is small.

Let the distance between the two receivers be denoted by $AB = l$ and let the angle between the line joining them AB and the direction of the source of sound be denoted by ϕ. Let the pressure variation due to the incident sound at A be given by

$$A\epsilon^{i(pt-\frac{1}{2}kl\cos\phi)}$$

and that at B by

$$A\epsilon^{i(pt+\frac{1}{2}kl\cos\phi)}.$$

Let $\xi = P\epsilon^{ipt}$ denote the displacement of the surface of the sphere at A and $\eta = Q\epsilon^{ipt}$ denote the displacement of the surface of the sphere at B. Also let the velocity potential of the sound in the tube leading from A be denoted by

$$G\epsilon^{i(pt-k_1x)}$$

and that of the sound in the other tube by

$$H\epsilon^{i(pt-k_1x)}$$

then, as in the case of the single receiver, we have

$$\frac{P}{s}(\mu - mp^2 + Rip) - p^2aP\epsilon^{-ika} = -A\epsilon^{-i\Delta} + Q\frac{p^2a^2}{l}\epsilon^{-ikl} - \rho_1Gip,$$

$$\frac{Q}{s}(\mu - mp^2 + Rip) - p^2aQ\epsilon^{-ika} = -A\epsilon^{i\Delta} + P\frac{p^2a^2}{l}\epsilon^{-ikl} - \rho_1Hip,$$

where Δ has been written for $\frac{1}{2}kl\cos\phi$ and the density of water has been put equal to unity. Also we have

$$\xi = \frac{\alpha G\epsilon^{ipt}}{sv_1} \qquad \text{and} \qquad \eta = \frac{\alpha H}{sv_1}\epsilon^{ipt}$$

so that $P = \alpha G/sv_1$ and $Q = \alpha H/sv_1$ as in the case of a single receiver. Substituting these values of P and Q and solving the two equations for G and H we obtain:

$$H = -A\left\{\frac{\cos\Delta}{K-J} + \frac{i\sin\Delta}{K+J}\right\},$$

$$G = -A\left\{\frac{\cos\Delta}{K-J} - \frac{i\sin\Delta}{K+J}\right\},$$

where

$$K = \frac{\alpha}{s^2v_1}(\mu - mp^2 + Rip - p^2as\epsilon^{-ika}) + \rho_1ip$$

and

$$J = \frac{\alpha p^2a^2\epsilon^{-ikl}}{sv_1l}.$$

Let

$$M = m + as + (sa^2\cos kl)/l$$

and

$$S = R + \frac{p^2a^2s}{v} + \frac{s^2v_1\rho_1}{\alpha} + \frac{spa^2}{l}\sin kl,$$

so that

$$K - J = \frac{\alpha}{s^2v_1}(\mu - Mp^2 + ipS).$$

Also let

$$N = m + as - (sa^2 \cos kl)/l$$

and

$$T = R + \frac{p^2a^2s}{v} + \frac{s^2\rho_1v_1}{\alpha} - \frac{spa^2 \sin kl}{l},$$

so that

$$K + J = \frac{\alpha}{s^2v_1}(\mu - Np^2 + ipT).$$

Now let

$$\frac{\alpha}{s^2v_1}(\mu - Mp^2) = h \qquad \text{and let} \qquad \frac{\alpha}{s^2v_1}pS = g.$$

Also let

$$\frac{\alpha}{s^2v_1}(\mu - Np^2) = u \qquad \text{and let} \qquad \frac{\alpha}{s^2v_1}pT = w.$$

Hence

$$H = -A\left\{\frac{\cos\Delta}{h + ig} + \frac{i\sin\Delta}{u + iw}\right\},$$

$$G = -A\left\{\frac{\cos\Delta}{h + ig} - \frac{i\sin\Delta}{u + iw}\right\}.$$

These equations give

$$H = -A\epsilon^{i\psi}\sqrt{\frac{\cos^2\Delta}{h^2 + g^2} + \frac{\sin^2\Delta}{u^2 + w^2} - \frac{\sin 2\Delta\cdot(gu - hw)}{(h^2 + g^2)(u^2 + w^2)}},$$

$$G = -A\epsilon^{i\psi'}\sqrt{\frac{\cos^2\Delta}{h^2 + g^2} + \frac{\sin^2\Delta}{u^2 + w^2} + \frac{\sin 2\Delta\cdot(gu - hw)}{(h^2 + g^2)(u^2 + w^2)}},$$

where

$$\tan\psi = \frac{(u^2 + w^2)g\cos\Delta - (h^2 + g^2)u\sin\Delta}{(u^2 + w^2)h\cos\Delta + (h^2 + g^2)w\sin\Delta}$$

and

$$\tan\psi' = \frac{(u^2 + w^2)g\cos\Delta + (h^2 + g^2)u\sin\Delta}{(u^2 + w^2)h\cos\Delta - (h^2 + g^2)w\sin\Delta}.$$

Hence

$$\tan(\psi' - \psi) = \frac{2(hu + gw)\tan\Delta}{u^2 + w^2 - (h^2 + g^2)\tan^2\Delta}.$$

Thus there are two resonance frequencies approximately given by $h = 0$ and $u = 0$ but unless l is very small they almost coincide.

The determination of the direction of the incident sound requires that when $\Delta = 0$ then $\psi' - \psi = 0$ and that, for small values of Δ,

$\psi' - \psi$ shall be of the same sign as Δ. The formula for $\tan(\psi' - \psi)$ shows that when $\Delta = 0$ then $\psi' - \psi = 0$ in all cases, whatever the distance between the receivers. When Δ is small, so that $(h^2 + g^2)\tan^2\Delta$ can be neglected, $\psi' - \psi$ will have the same sign as Δ provided $hu + gw$ is positive. Now $hu + gw$ is equal to

$$\left(\frac{\alpha}{s^2 v_1}\right)^2 (\mu - Mp^2)(\mu - Np^2) + \left(\frac{\alpha}{s^2 v_1}\right)^2 p^2 ST,$$

which might be negative if p were between $\sqrt{\mu/M}$ and $\sqrt{\mu/N}$. When l is not very small M and N are nearly equal so that when p is between $\sqrt{\mu/M}$ and $\sqrt{\mu/N}$ then both $\mu - Mp^2$ and $\mu - Np^2$ are very small so that $hu + gw$ can not be negative unless ST is very small which is never the case. Consequently unless l is very small, which is never the case in practice, the conditions for giving correct direction are always satisfied.

When Δ is not very small the sound will all appear to come from the same direction provided $\psi' - \psi = 2\Delta$ for all frequencies audible. This will be the case if $h = u$ and $g = w$, for then

$$\tan(\psi' - \psi) = \frac{2\tan\Delta}{1 - \tan^2\Delta} = \tan 2\Delta.$$

If l is large compared with the radius a of the receivers then the two resonance frequencies are near together so that for frequencies near them, both h and u will be small. In this case approximately

$$\tan(\psi' - \psi) = \frac{2gw\tan\Delta}{w^2 - g^2\tan^2\Delta},$$

which is equal to $\tan 2\Delta$ if $g = w$. Now

$$g - w = \frac{\alpha}{s^2 v_1} p(S - T) = \frac{\alpha p^2}{2\pi v_1 l}\sin kl,$$

which is independent of the size of the receivers. If $\sin kl = 0$ or $kl = \pi$ then $g = w$ so that if $l = \lambda/2$ where λ is the wave-length in the water for the resonance frequency, we shall have $\psi' - \psi = 2\Delta$ approximately for frequencies near the resonance frequency and the greater part of the sound heard will then appear to all come from the same direction whatever the value of Δ.

It appears that the best value of l is equal to half the wave-length in the water for the resonance frequency of the receivers.

3. The Theory of Multiple Receivers.

In order to obtain more sound than can be got from a single receiver a number of receivers can be used connected by tubes to a single tube leading to the ear. The use of such multiple receivers was proposed in

June, 1917, by Professor Zeleny and the writer, who worked out the theory according to which receivers should be connected together and also showed experimentally that more sound could be obtained from multiple receivers than from single ones.

Multiple receivers designed according to the principles worked out by Professor Zeleny and the writer have since been employed extensively in anti-submarine work by the United States Navy.

Suppose a number of tubes with sections $\alpha_1\alpha_2\alpha_3\cdots\alpha_n$ are joined into a single tube of section γ. Let the velocity potential in the mth tube be

$$A_m\epsilon^{i(pt-kx)} + B_m\epsilon^{i(pt+kx)}$$

and in the single tube

$$C\epsilon^{i(pt-kx)} + D\epsilon^{i(pt+kx)}.$$

Here x is the distance measured along the tubes and we shall suppose that x has the same value x_1 at the junction in all the potentials.

Then since the pressure at the junction has the same value in all the tubes, we must have

$$A_m\epsilon^{-ikx_1} + B_m\epsilon^{ikx_1}$$

equal to a constant P say, whatever the value of m. Also the total flow of air into the junction along the n tubes must be equal to the flow out along the single tube so that

$$\sum_1^n \alpha_m(A_m\epsilon^{-ikx_1} - B_m\epsilon^{ikx_1}) = \gamma(C\epsilon^{-ikx_1} - D\epsilon^{ikx_1}).$$

Putting $B_m\epsilon^{ikx_1} = P - A_m\epsilon^{-ikx_1}$ this gives

$$\Sigma\alpha_m(2A_m\epsilon^{-ikx_1} - P) = \gamma(2C\epsilon^{-ikx_1} - P).$$

Hence we see that if $\gamma = \Sigma\alpha_m$ then

$$C = (\Sigma\alpha_m A_m)/\gamma$$

and in the same way if $\gamma = \Sigma\alpha_m$ then

$$D = (\Sigma\alpha_m B_m)/\gamma.$$

Suppose a multiple receiver is made up of n similar receivers each connected to a tube of section α and let the potential in the mth tube be $A_m\epsilon^{i(pt-kx)} + B_m\epsilon^{i(pt+kx)}$. Let the n tubes be joined together, in any way, into a single tube of section γ in which the potential is $C\epsilon^{i(pt-kx)}$. Then if at all junctions between two or more tubes the total cross section is conserved we shall have $\gamma = n\alpha$,

$$C = \frac{\Sigma\alpha_m A_m}{\gamma} = \frac{\Sigma A_m}{n},$$

and

$$\Sigma B_m = 0.$$

Also if the A's are all equal then $C = A$. In this case all the sound comes out along the single tube.[1]

The A's will all be equal if all the receivers produce equal amplitudes of vibration and if the phases of the vibrations are all the same at places where x has the same value. This requires that if the phase at the mth receiver is θ_m and x_m denotes the value of x at this receiver then $\theta_m + kx_m$ must have the same value for all the receivers and if all the receivers vibrate with equal phases then x_m must have the same value for all the receivers.

It appears that the sounds from any number of equal receivers can all be combined into a single tube if when two or more tubes are joined the area of cross section is conserved and if the paths from each receiver to the final tube are such that $\theta_m + kx_m$ has the same value for all the receivers.

Since sharp bends in a tube may produce some reflection, the tubes should also be as nearly straight as possible. If desired, the sound can be concentrated into a narrower tube by means of a cone provided the cross section is not changed much in a length equal to the wave-length of the sound.

The theory of a multiple receiver designed in accordance with the above principles will now be considered. Let there be n equal receivers each connected to a tube of section α and let the n tubes be combined into a single tube of section $n\alpha$. To simplify the theory it will be supposed that the paths from each receiver to the final tube are all of the same length. The receivers will also be supposed to be arranged in a symmetrical way so that each one bears the same relation to the others; for example this would be the case if the receivers were arranged on a circle at equidistant points.

Each receiver will be supposed to be an elastic sphere of small internal volume and the notation used will be the same as in the previous sections of this paper.

Let ξ_m denote the radial displacement of the surface of the mth sphere and then we have

$$m\ddot{\xi}_m + R\dot{\xi}_m + \mu\xi_m = - s\delta P, \tag{1}$$

where δP denotes the total pressure variation on the surface of the sphere. This pressure variation δP is made up of (1) the pressure due to the source, (2) the pressure due to the sound emitted by the mth sphere, (3) that due to the sound from the other spheres, (4) that due to the sound in the air tube. (1) will be denoted by $S\epsilon^{i(pt+\theta_m)}$, (2) is equal

[1] See Theory of Sound, Rayleigh, Vol. II., p. 63.

to $-\rho a p^2 \xi_m \epsilon^{-ika}$, (3) is equal to $-\rho a^2 p^2 \Sigma \frac{\xi \epsilon^{-ikr}}{r}$, in which there is one term for each of the other spheres ξ being the radial displacement and r the distance from the mth sphere measured between centers. (4) is equal to $\rho_1 ip(A_m + B_m)\epsilon^{ipt}$. (2) and (3) can be combined into $-\rho a^2 p^2 \sum_1^n \frac{\xi \epsilon^{-ikr}}{r}$, in which for the mth sphere $r = a$ and $\xi = \xi_m$.

Hence equation (1) becomes

$$m\ddot{\xi}_m + R\dot{\xi}_m + \mu\xi_m = -s\left\{ S\epsilon^{i(pt+\theta_m)} - \rho p^2 a^2 \sum_1^n \frac{\xi\epsilon^{-ikr}}{r} + \rho_1 ip(A_m + B_m)\epsilon^{ipt} \right\}. \quad (2)$$

We have

$$s\dot{\xi}_m = \alpha i k_1 (A_m - B_m)\epsilon^{ipt},$$

so that

$$\xi_m = \frac{\alpha}{s v_1}(A_m - B_m)\epsilon^{ipt}.$$

Putting this value of ξ_m in (2) and adding up the n equations like (2) we get, after some reduction remembering that $\Sigma B_m = 0$ and that owing to the assumed symmetry

$$\sum_1^n \sum_1^n \frac{\xi\epsilon^{-ikr}}{r} = \sum_1^n \xi_m \sum_1^n \frac{\epsilon^{-ikr}}{r},$$

$$\Sigma A_m = \frac{-s^2 v_1 n S \Sigma \epsilon^{i\theta_m}}{\alpha\left(\mu - mp^2 + Rip + \frac{is^2\rho_1 p v_1}{\alpha} - \frac{s^2\rho p^2}{4\pi}\sum_1^n \frac{\epsilon^{-ikr}}{r}\right)}.$$

In what follows it will be supposed that all the receivers are equally distant from the source in the water so that

$$\sum_1^n \epsilon^{i\theta_m} = n.$$

If the potential in the tube leading from the multiple receiver is $C\epsilon^{i(pt-k_1x)}$ then as we have seen

$$C = \frac{\Sigma A_m}{n}.$$

The amount of sound energy E passing along this tube in unit time is $\frac{1}{2}\rho_1 v_1 k_1^2 C^2 \alpha n$, so that

$$E = \frac{\rho_1 v_1 s^4 p^2 S^2 n}{2\alpha\left\{\left(\mu - mp^2 - \frac{s^2\rho p^2}{4\pi}\Sigma\frac{\cos kr}{r}\right)^2 + p^2\left(R + \frac{s^2\rho_1 v_1}{\alpha} + \frac{s^2\rho p}{4\pi}\Sigma\frac{\sin kr}{r}\right)^2\right\}}$$

when $n = 1$ this reduces to the expression previously found for a single receiver since

$$\frac{\cos ka}{a} = \frac{1}{a} \quad \text{and} \quad \frac{\sin ka}{a} = k.$$

In the case of rubber or brass receivers it is probable that we may put $R = 0$. The expression for E shows that, for a certain value of p, E has a maximum value. Usually this maximum will occur near to the value of p given by

$$\mu - mp^2 - \frac{s^2\rho p^2}{4\pi}\Sigma\frac{\cos kr}{r} = 0,$$

which will be denoted by p_1. For a single receiver p_1 is given by

$$\mu - mp^2 - s\rho a p^2 = 0$$

so that we see that p_1 for a multiple receiver is smaller than for a single receiver. This result agrees with observations. When $p = p_1$

$$E = \frac{\rho_1 v_1 S n}{2\alpha\left(\frac{\rho_1 v_1}{\alpha} + \frac{\rho p}{4\pi}\Sigma\frac{\sin kr}{r}\right)^2}.$$

This is a maximum with respect to α when

$$\alpha = \frac{4\pi\rho_1 v_1}{\rho p_1 \Sigma\frac{\sin kr}{r}}.$$

When α has this value which we will denote by α_1 and $p = p_1$ then

$$E = \frac{S^2 n\alpha}{8\rho_1 v_1} = \frac{\pi S^2 n}{2\rho p_1 \Sigma\frac{\sin kr}{r}}.$$

If all the receivers are rather near together so that kr is small then

$$\Sigma\frac{\sin kr}{r} = nk,$$

approximately and

$$\Sigma\frac{\cos kr}{r} = \frac{1}{a} + \frac{n-1}{\bar{r}},$$

where $(\bar{r})^{-1}$ denotes the mean value of r^{-1}. Hence in this case when $R = 0$

$$E = \frac{\frac{1}{2}\rho_1 v_1 s^4 p^2 S^2 n\alpha}{\alpha^2\left\{\mu - p^2\left[m + s\rho a\left(1 + \frac{(n-1)a}{\bar{r}}\right)\right]\right\}^2 + p^2\left\{s^2\rho_1 v_1 + \frac{s^2\rho p^2 n\alpha}{4\pi v}\right\}^2}.$$

This expression is of the same form as that previously found for a single receiver so that it appears that a multiple receiver in which the distances between the receivers are small compared with the wave-length in the water, can theoretically be replaced by an equivalent single receiver.

Let $\alpha' s' \mu' m' p_1'$ denote the values of these quantities for the equivalent single receiver. Then we must have $\alpha' = n\alpha$, $\mu' = \mu s'^2/ns^2$ and $p_1' = p_1$ for if these equations are true the expression for E, just given, reduces to that for a single receiver. These three equations contain only three unknown quantities because μ m and s are determined in terms of the elastic properties and density of the sphere when its dimensions are known.

At the resonance frequency $p = p_1$ and

$$E = \frac{\frac{1}{2}\rho_1 v_1 S^2 n\alpha}{\left(\rho_1 v_1 + \dfrac{\rho p_1^2 n\alpha}{4\pi v}\right)^2}.$$

This shows that at the resonance frequency a multiple receiver, in which the receivers are rather near together, can be replaced by any single receiver for which $\alpha' = n\alpha$ and which has the same resonance frequency as the multiple receiver. If in addition $ns^2\mu' = s'^2\mu$ the single receiver and the multiple receiver will be equivalent at all frequencies.

If κ denotes the bulk modulus of elasticity and ν the rigidity modulus of the material of which the receivers are made then

$$\mu = \frac{48\pi\kappa\nu\tau}{(\kappa + \frac{4}{3}\nu)},$$

where τ is the thickness of the hollow sphere. Let

$$\frac{48\pi\kappa\nu}{\kappa + \frac{4}{3}\nu} = C$$

so that $\mu = C\tau$. Let δ denote the density of the material of the receivers. Then the equation $p_1' = p_1$ gives

$$\frac{\mu'}{m' + s'\rho a'} = \frac{\mu}{m + s\rho a(1 + (n-1)a/\bar{r})},$$

where

$$m' = 4\pi a'^2 \delta' \tau'; \qquad m = 4\pi a^2 \delta \tau$$

$$\mu' = C'\tau'; \qquad \mu = C\tau.$$

The equation $ns^2\mu' = \mu s'^2$ gives

$$\tau' = \frac{a'^4 \tau C}{na^4 C'},$$

so that we get

$$\frac{C'}{C} = \frac{a'^3\delta'}{a^2\{a'(\delta + a/\tau + a^2(n-1)/\tau\bar{r} - na^2/\tau\}}. \qquad (3)$$

We see from this that a' must be greater than

$$na^2/(\tau\delta + a + a^2(n - 1)/\bar{r}).$$

If a' is taken big enough to satisfy this condition then if a material can be found for which C'/δ' has the value given by (3) it will be possible to construct the equivalent single receiver out of it.

The equivalent single receiver can not be made of the same material as the multiple receiver unless the equation in a'

$$a^2\{a'(\delta + a/\tau + a^2(n - 1)/\tau\bar{r}) - na^2/\tau\} = a'^3\delta$$

has positive real roots which is not the case in practical cases.

For example if $n = 8$, $a = 1$ cm., $\tau = 0.1$ cm., $\bar{r} = 14$ cms., $\delta = 1$ this equation becomes

$$a'^3 = 16a' - 80,$$

which has no positive real roots so that in this case a single equivalent receiver can not be made out of the same material as the multiple receiver.

But if we take $a' = 7$ cms. we get $C'/C = 10\cdot 7\delta'$, so that a single equivalent receiver of radius 7 cms. could be made out of a material for which $C'/\delta' = 10\cdot 7C$. If $\delta' = 1$ this gives $\tau' = 2.8$ cms. If a material were available for which $C' = 1{,}000C$ and $\delta' = 10$ then we have

$$100\{a'(1 + 10 + 5) - 80\} = a'^3$$

which has a positive root $a' = 5.08$ cms. approximately. This gives $\tau' = 0.00832$ cm. A receiver with a radius of 5 cms. would have too large an internal volume for this to be neglected but a solid core could be put inside it so as to make its effective volume small. A receiver of 5 cms. radius and thickness only 0.00832 cm. would be too thin to be useful for practical purposes.

If the receivers in the multiple receiver are made of rubber then C is very small compared to its value for other substances so that it is then practically impossible to make a single equivalent receiver because when C'/C is large the thickness of the single receiver becomes very small unless its radius is very large. This follows from $\tau' = \tau a'^4 C/na^4 C'$.

It appears that single equivalent receivers can not be made for practical purposes. This conclusion agrees with experience, for a great many attempts were made to make single receivers equivalent to a multiple rubber receiver without any success.

If for a multiple receiver $n = 10$, $a = 3$ cms., $m = 100$ grams, $\bar{r} = 15$ cms. and the resonance frequency $N_1 = 328$, the equation for E/S^2 when α has the best value at the resonance frequency becomes

$$\frac{E}{S^2} = \frac{0.213}{51.5\left(\frac{1 - f^2}{f}\right)^2 + (1 + f^2)^2},$$

where $f = N/N_1$.

For a single receiver equal to each of the 10 receivers in the case just given we get $N_1 = 478$ and when α again has the best value for this frequency

$$\frac{E}{S^2} = \frac{0.100}{536\left(\frac{1 - f^2}{f}\right)^2 + (1 + f^2)^2}.$$

Fig. 5 shows the values of E/S^2 given by these equations for different values of N.

It appears that the multiple receiver is more sensitive than the single

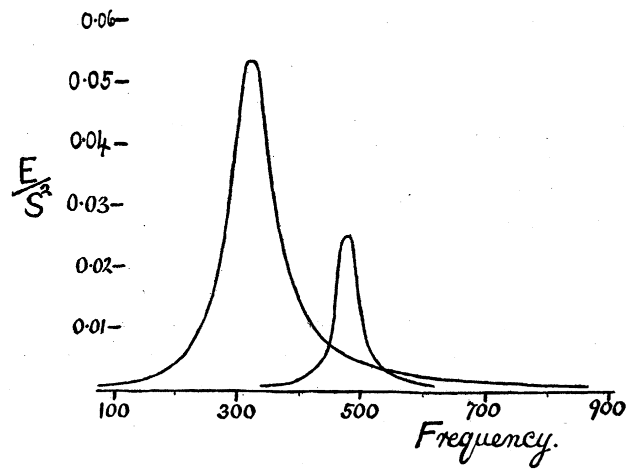

Fig. 5.

one except very near the resonance frequency of the single receiver. The single receiver is also of higher pitch and is more sharply resonant.

As one more example take a multiple receiver for which $n = 8$, $a = 1$ cm., $\delta = 1$, $\tau = 0.1$ cm., $p_1 = 5{,}000$, $\alpha = 3/8$ sq. cm. These values give

$$113\frac{E}{S^2} = \frac{4}{32.2\left(\frac{1 - f^2}{f}\right)^2 + (1 + f^2)^2}.$$

Also take a single receiver for which $a' = 2$, $\alpha' = 3$, $\delta' = 1$, $\tau' = 2/3$ cm. and $p_1' = 5{,}000$. This single receiver has the same resonance frequency and the same sized tube as the multiple receiver and it is made of the same material but it does not satisfy the equation $ns^2\mu' = \mu s'^2$. For this receiver

$$113\frac{E}{S^2} = \frac{4}{354\left(\frac{1 - f^2}{f}\right)^2 + (1 + f^2)^2}.$$

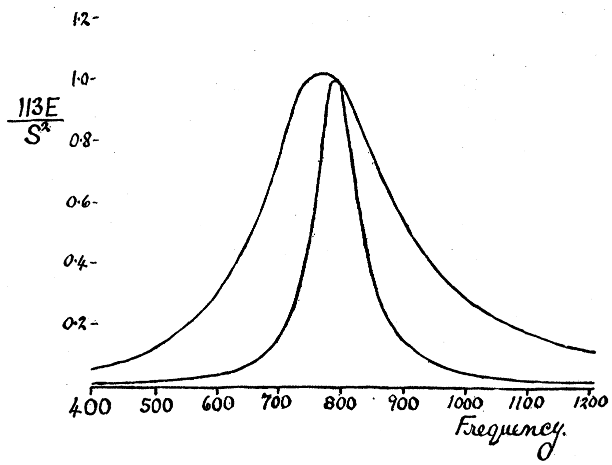

Fig. 6.

The relation between $113E/S^2$ and N for these two receivers is shown in Fig. 6. It appears that the multiple receiver is more sensitive except at the frequency $N_1 = p_1/2\pi$ where they are equally sensitive.

4. Lines of Receivers.

The theory of a number of receivers, either single or multiple, arranged with uniform spacing along a straight line will now be considered. This arrangement was proposed by Professor Max Mason early in June, 1917, and has been extensively employed by the United States Navy during the war.

If the distance from any receiver to the next one in the line is d and if the angle between the line and the direction of the incident sound is ϕ then the distance the sound has to travel in the water increases, as we go along the line from one receiver to the next one, by $d \cos \phi$. If v is the velocity of sound in water the time difference from one receiver to the next one is therefore $(d \cos \phi)/v$. The receivers are connected by tubes to a single tube leading to the ear. In order that the sound from all the receivers may arrive at the ear at the same instant it is necessary that the tubes be of different lengths. The difference of length for two adjacent receivers must be $(v_1 d \cos \phi)/v$, where v_1 is the velocity of sound in air. Thus as ϕ varies the differences between the lengths of successive tubes must be varied in order that the sounds from all the receivers may always arrive together at the ear so as to produce the maximum possible intensity. Instruments for producing this variation were devised by Professor Mason and are called compensators.

It will be assumed here that the tube lengths are such that the sounds from all the receivers arrive together at the ear. In this case for a very long line the angle of incidence of the external sound makes no difference so long as the intensity of the external sound at each receiver remains unchanged. When two or more tubes are connected together the area of cross section must be conserved as in the multiple receivers considered in the previous section.

The theory of the multiple receiver can be applied to the case of a long line of receivers equally spaced for this theory applies to receivers equally spaced round the circumference of a circle and if we suppose the diameter of the circle to be indefinitely increased the sound energy from each receiver will be nearly the same as from each receiver in a long line.

In this case we have

$$\Sigma\frac{\sin kr}{r} = k + 2\left\{\frac{\sin kd}{d} + \frac{\sin 2kd}{2d} + \cdots\right\}$$

$$= \pi/d \text{ when } d < \lambda.$$

$$= k \text{ when } d \text{ is very large.}$$

Also

$$\Sigma\frac{\cos kr}{r} = \frac{1}{a} + 2\left\{\frac{\cos kd}{d} + \frac{\cos 2kd}{2d} + \cdots\right\}$$

$$= \frac{1}{a} - \frac{1}{d}\log\left(4\sin^2\left(\frac{kd}{2}\right)\right).$$

Hence the sound energy from a length l of the line when d is less than λ will be given by

$$E = \frac{\rho_1 v_1 s^4 p^2 S^2 l}{2\alpha d\left\{\left[\mu - mp^2 - s\rho a p^2\left(1 - \frac{a}{d}\log\left(4\sin^2\left(\frac{kd}{2}\right)\right)\right)\right]^2 + p^2\left[\frac{s^2\rho_1 v_1}{\alpha} + \frac{s^2\rho p}{4d}\right]^2\right\}}.$$

At the resonance frequency this becomes

$$E = \frac{\rho_1 v_1 \alpha S^2 l}{2d(\rho_1 v_1 + \rho p_1 \alpha/4d)^2}.$$

This is a maximum with respect to α/d when

$$\alpha/d = 4\rho_1 v_1/\rho p_1 = 27/N_1,$$

where

$$N_1 = p_1/2\pi.$$

When α/d has this value then

$$E = \frac{S^2 l}{2\rho p_1} = \frac{S^2 l}{4\pi\rho N_1}.$$

If β denotes the area of the incident wave fronts through which an amount of energy equal to E flows then at the resonance frequency when $\alpha/d = 4\rho_1 v_1/\rho p_1$

$$E = \frac{1}{2}\frac{S^2\beta}{\rho v} = \frac{S^2 l}{4\pi\rho N_1},$$

so that

$$\frac{\beta}{l} = \frac{v}{2\pi N_1} = \frac{\lambda_1}{2\pi},$$

where λ_1 is the wave-length in the water for the resonance frequency.

We may observe that if, at the resonance frequency $\sin(kd/2) = \frac{1}{2}$ or $d = \lambda_1/6$ then $\log(4\sin^2(kd/2)) = 0$, so that $p_1^2 = \mu/(m + s\rho a)$ as for a single receiver.

If d is less than $\lambda_1/6$ the resonance frequency will be lower than for a single receiver. If $d = \lambda_1/6$ then the equation $\alpha/d = 27/N_1$ gives

$$\alpha = \frac{9\lambda_1}{2N_1} = \frac{9}{2}\frac{\lambda_1^2}{v} = \frac{\lambda_1^2}{32000}.$$

5. Receivers Distributed over Large Areas.

Instruments have been proposed in which a large number of sound receivers was to be uniformly distributed over a considerable area on the side of a ship. If the area covered has linear dimensions greater than the wave length of the sound in the water then the sound energy transmitted per unit area may be calculated approximately by assuming that the area covered with receivers is an infinite plane on which plane waves are incident. It will be convenient to begin by considering a simple ideal case.

Consider an infinite plane completely covered with thin flexible diaphragms each of area β. Suppose that behind each diaphragm there

Fig. 7.

is a small enclosed space of volume V from which a tube of section α leads. This arrangement is shown in Fig. 7. Suppose that the space on the right of the plane is filled with water and that the cavities and tubes are filled with air and that a train of sound waves in the water is travelling towards the plane.

Let the velocity potential of the incident sound be given by

$$\phi = \phi_1 \epsilon^{i(ax+by+ct)},$$

where x denotes the distance from the infinite plane.[1] Let the potential of the reflected waves in the water be given by

$$\phi' = \phi_2 \epsilon^{i(-ax+by+ct)}$$

and let the potential in the tubes leading from the receivers be denoted by

$$\phi'' = \phi_3 \epsilon^{i(a_1 x+by+ct)}.$$

In the tubes $(-x)$ denotes the distance from the receivers measured along the tubes.

Let us suppose that the diaphragms are massless and perfectly flexible so that the air pressure in the cavities must be equal to the pressure in the water. In this case we have at $x = 0$

$$-\rho_1 \frac{d\phi''}{dt} = -\rho\left(\frac{d\phi}{dt} + \frac{d\phi'}{dt}\right)$$

where ρ_1 = density of air and ρ = density of water. Also, if the volume V is very small, we may put the flow of air along the tubes at $x = 0$ equal to the flow of water over the plane $x = 0$.

Hence at $x = 0$

$$\beta\left(\frac{d\phi}{dx} + \frac{d\phi'}{dx}\right) = \alpha\frac{d\phi''}{dx}.$$

These equations give

$$\rho_1\phi_3 = \rho(\phi_1 + \phi_2)$$

and

$$\alpha a_1 \phi_3 = \beta a(\phi_1 - \phi_2),$$

so that

$$1 - \left(\frac{\phi_2}{\phi_1}\right)^2 = \frac{4\dfrac{\rho\alpha a_1}{\rho_1\beta a}}{\left\{1 + \dfrac{\rho\alpha a_1}{\rho_1\beta a}\right\}^2}.$$

Now $(\phi_2/\phi_1)^2$ is the fraction of the incident sound energy which is reflected so that $1 - (\phi_2/\phi_1)^2$ is equal to the fraction of the incident sound energy which is transmitted from the water into the air in the tubes. If $\rho\alpha a_1/\rho_1\beta a = 1$ then $1 - (\phi_2/\phi_1)^2 = 1$ so that all the energy is transmitted and there is no reflected sound.

If θ denotes the angle of incidence then $a = (2\pi \cos\theta)/\lambda$ and $a_1 = (2\pi\cos\theta_1)/\lambda_1$ where λ is the wave-length in water and λ_1 that in air. Also θ_1 is given by $v/v_1 = \sin\theta/\sin\theta_1$ where v denotes the velocity of sound in water and v_1 that in air. Hence for complete transmission

[1] See Theory of Sound, Rayleigh, Vol. II., p. 79.

we must have

$$\alpha = \beta \frac{\rho_1 v_1 \cos \theta}{\rho v \cos \theta_1}.$$

When $\theta = 0$ this gives $\alpha = \beta/3390$ approximately and when $\theta = \pi/2$ it gives $\alpha = 0$.

It appears that, theoretically, complete transmission can be secured for any given angle of incidence by giving the proper value to α/β.

The difference between the properties of water and air can be compensated by concentrating the sound into a much smaller area in the air by means of the tubes from each receiver.

If the tubes and enclosed spaced are omitted, we get $\alpha = \beta$ so that

$$1 - (\phi_2/\phi_1)^2 = \frac{4\dfrac{\rho a_1}{\rho_1 a}}{\left\{1 + \dfrac{\rho a_1}{\rho_1 a}\right\}^2},$$

which is equal to the fraction of the sound transmitted from water to air when there is nothing in between them. This fraction is only about 0.12 per cent. at normal incidence.

Now suppose that the diaphragms are elastic so that if ξ denotes the mean displacement of a diaphragm in the x direction then at $x = 0$

$$\frac{d\phi}{dx} + \frac{d\phi'}{dx} = \dot{\xi} \tag{1}$$

and

$$\delta p + \rho\left(\frac{d\phi}{dt} + \frac{d\phi'}{dt}\right) = \mu\xi + m\ddot{\xi}, \tag{2}$$

where δp denotes the variation of the air pressure in the cavity and μ denotes the pressure required to produce unit value of ξ and m denotes the effective mass of the diaphragms.

Let η denote the air displacement in the tubes at $x = 0$ so that

$$\delta p = -\gamma p \delta V/V = -\gamma p(\xi\beta - \alpha\eta), \tag{3}$$

where γ is the ratio of the specific heats of air. Also at $x = 0$

$$-\rho_1 \frac{d\phi''}{dt} = \delta p \tag{4}$$

and

$$\frac{d\phi''}{dx} = \dot{\eta}. \tag{5}$$

These five equations, after eliminating ξ, η, δp and ϕ_3 give

$$1 - \left(\frac{\phi_2}{\phi_1}\right)^2 = \frac{4\alpha\beta\rho\rho_1 a a_1}{\{\alpha a_1\rho + \beta a\rho_1 + aV(\mu - mc^2)/v_1^2\}^2 + \{V\rho c^2/v_1^2 - \alpha a a_1(\mu - mc^2)/c^2\}^2}.$$

If $\mu = 0$, $m = 0$ and $V = 0$ this reduces to the expression previously obtained.

In the case of normal incidence we have $a = c/v$ and $a_1 = c/v_1$ so that

$$1 - \left(\frac{\phi_2}{\phi_1}\right)^2 = \frac{4\alpha\beta\rho\rho_1 vv_1}{\{\alpha\rho v + \beta\rho_1 v_1 + V(\mu - mc^2)/v_1\}^2 + \{V\rho cv/v_1 - \alpha(\mu - mc^2)/c\}^2}.$$

If V is so small that the terms containing it can be neglected this expression has a maximum value with respect to c when $c = \sqrt{\mu/m}$ which corresponds to the natural frequency of the diaphragms when vibrating freely. When $c = \sqrt{\mu/m}$ and V can be neglected the expression is again the same as that previously obtained. It appears that if $\alpha = \beta\rho_1 v_1/\rho v$ then at the resonance frequency, when V is very small, all the sound will be transmitted but at all other frequencies some will be reflected.

It is not possible in practice to make V so small that it can be neglected so that complete transmission can not be secured. If V is not small enough to be neglected then the best value of α is greater than that given by $\alpha = \beta\rho_1 v_1/\rho v$.

As an example suppose $\beta = 50$ sq. cms., $m = 2$ gms., $\alpha = 1$ sq. cm., $V = 2$ c.c., $\mu = 10^7$. The following values of $1 - (\phi_2/\phi_1)^2$ are then obtained.

c	$1-(\phi_2/\phi_1)^2$
500	0.0560
1000	0.0577
5000	0.0534
10000	0.0419

If α is equal to 0.1 sq. cm. instead of 1 sq. cm. we get:

c	$1-(\phi_2/\phi_1)^2$
338	0.416
1000	0.344
5000	0.0589
10000	0.0162

Thus the smaller value of α gives better results at low frequencies but worse at high frequencies.

Instead of a large area covered with flexible diaphragms an area over

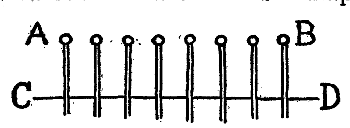

Fig. 8.

which small sound receivers are distributed uniformly may be used. Each receiver is connected to a tube and the plane containing the receivers is at a certain distance l from the side of the ship. This arrangement is shown in Fig. 8. AB is the plane containing the receivers and CD

the side of the ship. The side of the ship reflects almost perfectly the sound which is incident upon it. The reflecting surface is approximately the surface of the air inside the wall of the ship. To increase the effective distance between the layer of receivers and the air surface metal plates may be put in between. Such metal plates are nearly equivalent acoustically to an equal mass of water. It will be supposed that the length l is the thickness of water acoustically equivalent to the actual structure between the receivers and the reflecting air surface.

If the distance between adjacent receivers is small compared with the wave-length of the sound in the water and if the radius of a receiver is not very small compared with the distance between adjacent receivers then it will be approximately correct to suppose that when plane sound waves are incident the reflected waves are also plane. We have to consider five trains of plane waves. The incident waves, the reflected waves to the right of AB and the two sets of waves between AB and CD and the waves in the air tubes leading from the receivers.

Let the potential of the incident sound waves be given by

$$\phi_1' = \phi_1 \epsilon^{i(ax+by+ct)},$$

where x denotes the distance from the plane AB containing the receivers. Let the potential of the reflected sound to the right of AB be given by

$$\phi_2' = \phi_2 \epsilon^{i(-ax+by+ct)}.$$

Let the potentials of the two sets of waves in the water between AB and CD be given by

$$\phi_3' = \phi_3 \epsilon^{i(ax+by+ct)},$$

$$\phi_4' = \phi_4 \epsilon^{i(-ax+by+ct)}.$$

Let the potential of the sound in the air tubes be given by

$$\phi_5' = \phi_5 \epsilon^{i(a_1 x+by+ct)},$$

where $(-x)$ is the distance from the receivers measured along the tubes.

If we suppose that the volume of the receivers can be neglected then at $x = 0$ we may put the total flow of water towards the plane $x = 0$ equal to the flow of air into the air tubes from the receivers. Hence

$$-\frac{d\phi_1'}{dx} - \frac{d\phi_2'}{dx} + \frac{d\phi_3'}{dx} + \frac{d\phi_4'}{dx} = -f\frac{d\phi_5'}{dx},$$

where f denotes the ratio of the total cross section of the tubes to the area covered with receivers. Therefore

$$-\phi_1 + \phi_2 + \phi_3 - \phi_4 = -\phi_5 a_1 f/a. \qquad (1)$$

Also, since the pressure in the water at $x = 0$ has only one average value at any instant we get

$$\phi_1 + \phi_2 = \phi_3 + \phi_4. \qquad (2)$$

It will be supposed that the diminution of the volume of the receivers is proportional to the difference between the pressure in the water and the pressure in the air so that

$$\phi_1 + \phi_2 = \phi_5\left(\frac{\rho_1}{\rho} + \frac{i\mu a_1}{\rho c^2}\right), \qquad (3)$$

where ρ denotes the density of water ρ_1 that of air and μ is an elastic constant for the receivers equal to the pressure difference required to produce unit air displacement in the tubes at $x = 0$.

Also at $x = -l$ the total pressure variation is zero so that

$$\phi_4 = -\phi_3\epsilon^{-2ial}. \qquad (4)$$

Eliminating ϕ_3, ϕ_4 and ϕ_5 we obtain

$$\phi_2 = -\phi_1\frac{\left\{\frac{\rho_1 a}{\rho a_1} + \frac{\mu a}{\rho c^2}\tan al + i\left(\left(f - \frac{\rho_1 a}{\rho a_1}\right)\tan al + \frac{\mu a}{\rho c^2}\right)\right\}}{\left\{\frac{\rho_1 a}{\rho a_1} - \frac{\mu a}{\rho c^2}\tan al + i\left(\left(f + \frac{\rho_1 a}{\rho a_1}\right)\tan al + \frac{\mu a}{\rho c^2}\right)\right\}}.$$

If $\phi_2 = 0$ there is no reflected sound so that all the incident sound is transmitted into the air tubes. This will be the case provided

$$\rho_1 a/\rho a_1 + (\mu a \tan al)/\rho c^2 = 0$$

and

$$(f - \rho_1 a/\rho a_1)\tan al + \mu a/\rho c^2 = 0,$$

or if $\tan al = -\rho_1 c^2/\mu a_1$ and $f = \rho_1 a/\rho a_1 + a a_1 \mu^2/\rho\rho_1 c^2$.

The following table gives the values of l and f for which there is complete transmission for several values of μ and n for the case of normal incidence.

$\mu = 3.5 \times 10^5$.			$\mu = 7 \times 10^5$.			$\mu = 14 \times 10^5$.		
n.	*l*.	*f*.	*n*.	*l*.	*f*.	*n*.	*l*.	*f*.
1,500	34.4	0.00053	3,000	17.25	0.00053	3,000	19.75	0.00118
1,000	56.0	0.00080	1,500	39.4	0.00121	1,500	43.0	0.00379
			1,000	62.6	0.00227	1,000	67.0	0.00815

Here n = frequency and l is in cms.

If $\mu = 0$ so that the receivers are perfectly flexible then for complete transmission $\tan al = -\infty$ so that $al = \pi/2$, or $l = \lambda/4 \cos\theta$ where θ is the angle of incidence, and $f = \rho_1 a/\rho a_1$ which is equal to $\rho_1 v_1/\rho v$ in the case of normal incidence. Calculating $1 - (\phi_2/\phi_1)^2$ we get it equal to

$$\frac{4f\rho a_1/\rho_1 a}{\left\{1 + \left(\frac{\mu a_1}{\rho_1 c^2}\right)^2\right\}\cot^2 al + \frac{2\mu\rho a_1^2 f}{\rho_1^2 a c^2}\cot al + \left(\frac{\mu a_1}{\rho_1 c^2}\right)^2 + \left\{1 + f\frac{\rho a_1}{\rho_1 a}\right\}^2},$$

which in the case of normal incidence reduces to

$$\frac{4\gamma}{\left\{1 + \left(\frac{\alpha}{n}\right)^2\right\}\cot^2\phi + \frac{2\alpha\gamma}{n}\cot\phi + \left(\frac{\alpha}{n}\right)^2 + (1+\gamma)^2},$$

where $\gamma = f\rho v/\rho_1 v_1$, $\alpha = \mu/2\pi\rho_1 v_1$, $\phi = 2\pi nl/v$. This expression is equal to unity when $\tan\phi = -n/\alpha$ and $\gamma = 1 + \alpha^2/n^2$, which are, of course, equivalent in the case of normal incidence, to the two conditions previously obtained.

Fig. 9 shows the percentage of sound transmitted into the air tubes

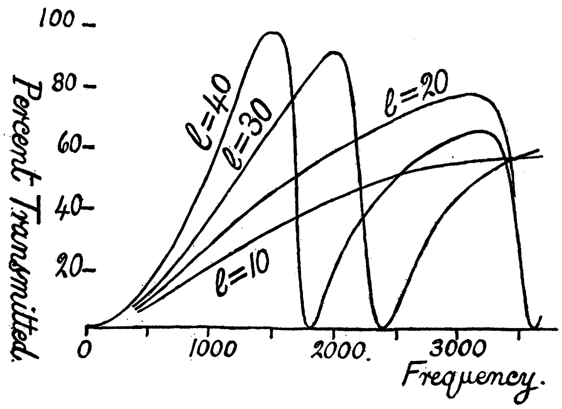

Fig. 9.

for different values of the frequency n and l in the case when $\mu = 7 \times 10^5$, $f = 0.00136$ and $\theta = 0$.

It will be seen that the maximum transmission occurs for a value of n which is about 0.85 of that for which the transmission is zero. Even with small values of l there is considerable transmission at high frequencies.

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